PLANE TRANSONIC GAS FLOWS WITH SINGULARITIES ON THE SONIC LINE

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1. Investigations of special cases of plane potential gas flows at transonic speeds lead to boundary-value problems of mixed elliptic-hyperbolic equations. In the most interesting cases of solutions of such problems, singular points occur on the sonic line. The determination of the character of such a singularity often forms the essential difficulty of the problem. Such solutions with singularities on the sonic line are known for the Tricomi equation

$$\eta \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0 \tag{1.1}$$

for which they form a class of self-similar solutions. However, plane potential gas flows are governed by the equation of Chaplygin, which can be replaced by the Tricomi equation near the sonic line only as an approximation. Chaplygin's equation does not possess self-similar solutions, a fact which complicates the integration of Chaplygin's equation with singularities on the sonic line.

In the present paper the desired solution is obtained in the form of an infinite series. The first term of the series corresponds to the self-similar solution of Tricomi's equation.

2. As Chaplygin showed, the equation of a plane, vorticity-free, adiabatic flow of a gas can be put in the form

$$4\tau^2 (1 - \tau) \frac{\partial^2 \psi}{\partial \tau^2} + 4\tau [1 + (\beta - 1)\tau] \frac{\partial \psi}{\partial \tau} + [1 - (2\beta + 1)\tau] \frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{2.1}$$

$$\left(\tau = \frac{v^2}{v_m^2},\ \beta = \frac{1}{\varkappa - 1},\ \tau = \tau_* = \frac{1}{2\beta + 1} \quad \text{for } v = a_*\right)$$

Here ψ represents the stream function, θ the angle between the velocity vector and an arbitrary direction, v the speed of the flow, v_m and v_* the maximal and the critical speed of the flow, respectively, and κ the adiabatic exponent.

From (2.1) it is clear that the sign of the coefficient of the last term changes when $\tau = \tau_*$. In this manner, when we study transonic gas flows, we have to deal with a mixed elliptic-hyperbolic differential equation. As Tricomi has shown [1], such equations are more readily investigated when they are transformed into a special canonical form, which facilitates the determination of the main term of the solution. For Equation (2.1) such a transformation was effected by Frankl [2] by means of a new variable η:

$$\eta = \left(\frac{3}{4}\int_{\tau}^{\tau_*}\sqrt{\frac{1-(2\beta+1)\tau}{1-\tau}}\,\frac{d\tau}{\tau}\right)^{2/3} \tag{2.2}$$

Equation (2.1) then takes on the form

$$\eta\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial\eta^2}+b(\eta)\frac{\partial\psi}{\partial\eta}=0 \tag{2.3}$$

$$b(\eta)=\frac{2\beta(2\beta+1)\tau^2\sqrt{\eta}}{[1-(2\beta+1)\tau]\sqrt{[1-(2\beta+1)\tau](1-\tau)}}-\frac{1}{2\eta} \tag{2.4}$$

In terms of an auxiliary variable z, Equations (2.2), (2.3) and (2.4) yield

$$z=\sqrt{[1-(2\beta+1)\tau]/(1-\tau)}.$$

$$b(\eta)=\frac{\kappa+1}{2}\frac{(1-z^2)^2}{z^3}\sqrt{\eta}-\frac{1}{2\eta}$$

$$\eta=\left[\frac{3}{4}\ln\left(\frac{h-z}{h+z}\right)^h\frac{1+z}{1-z}\right]^{2/3}\qquad\left(h=\sqrt{\frac{\kappa+1}{\kappa-1}}\right) \tag{2.5}$$

Equation (2.5) provides a parametric representation of the function $b(\eta)$ in terms of the parameter z. We note that for subsonic speeds, i.e. for $\tau < \tau_*$, the parameter z is a real number, and $\eta > 0$. For supersonic speeds z takes on purely imaginary values with $\eta < 0$, and, therefore, $b(\eta)$ remains real for arbitrary negative values of η. From (2.5) it is clear that in the neighborhood of $\eta = 0$ the function $b(\eta)$ can be developed in an infinite series

$$b(\eta)=b_0+b_1\eta+b_2\eta^2+\ldots \tag{2.6}$$

$$b_0=-\frac{2\kappa+5}{5(\kappa+1)^{1/3}},\qquad b_1=\frac{46\kappa^2+105\kappa+125}{175(\kappa+1)^{2/3}}$$

3. In order to find solutions of (2.3) with singular points on the sonic line $\eta = 0$, it is convenient to introduce still different variables

$$\rho = +\sqrt{\theta^2 + \tfrac{4}{9}\eta^3}, \qquad t = \frac{\theta}{\rho} \tag{3.1}$$

and to transform (2.3) into

$$(1-t^2)\frac{\partial^2\psi}{\partial t^2} - \frac{4}{3}t\frac{\partial\psi}{\partial t} + \rho^2\frac{\partial^2\psi}{\partial\rho^2} + \frac{4}{3}\rho\frac{\partial\psi}{\partial\rho} = -\frac{2}{3}\eta b(\eta)\left(t\frac{\partial\psi}{\partial t} - \rho\frac{\partial\psi}{\partial\rho}\right) \tag{3.2}$$

From (3.1) it follows that $\eta = (3\rho/2)^{2/3}(1-t^2)^{1/2}$, and hence

$$b(\eta) = \sum_{m=0}^{\infty} b_m\eta^m = \sum_{m=0}^{\infty} b_m\left(\frac{3}{2}\right)^{\frac{2}{3}m}\rho^{\frac{2}{3}m}(1-t^2)^{\frac{m}{3}}$$

Then Equation (3.2) can finally be expressed in the form

$$(1-t^2)\frac{\partial^2\psi}{\partial t^2} - \frac{4}{3}t\frac{\partial\psi}{\partial t} + \rho^2\frac{\partial^2\psi}{\partial\rho^2} + \frac{4}{3}\rho\frac{\partial\psi}{\partial\rho} =$$
$$= \left(t\frac{\partial\psi}{\partial t} - \rho\frac{\partial\psi}{\partial\rho}\right)\sum_{m=0}^{\infty} b_m\left(\frac{3}{2}\right)^{\frac{2m-1}{3}}\rho^{\frac{2m+2}{3}}(1-t^2)^{\frac{m+1}{3}} \tag{3.3}$$

We shall seek solutions of (3.3) in the form of the series

$$\psi(\rho, t) = \rho^\lambda f_0(t) + \rho^{\lambda+\frac{2}{3}}f_1(t) + \rho^{\lambda+\frac{4}{3}}f_2(t) + \ldots = \sum_{m=0}^{\infty}\rho^{\lambda+\frac{2}{3}m}f_m(t) \tag{3.4}$$

Substituting (3.4) into (3.3) and equating like powers of ρ, we obtain the recurrence relations for the determination of the coefficients $f_n(t)$ in (3.4):

$$(1-t^2)f_n'' - \tfrac{4}{3}tf_n' + \left(\lambda + \tfrac{2}{3}n\right)\left(\lambda + \tfrac{2}{3}n + \tfrac{1}{3}\right)f_n =$$
$$= \sum_{m=0}^{n-1} b_m\left(\frac{3}{2}\right)^{\frac{2m-1}{3}}(1-t^2)^{\frac{m+1}{3}}\left\{tf_{n-m-1}' - \left[\lambda + \tfrac{2}{3}(n-m-1)\right]f_{n-m-1}\right\}$$
$$(n = 0, 1, 2, \ldots) \tag{3.5}$$

The first relations are as follows:

$$(1-t^2)f_0'' - \tfrac{4}{3}tf_0' + \lambda\left(\lambda + \tfrac{1}{3}\right)f_0 = 0 \tag{3.6}$$

$$(1-t^2)f_1''-\tfrac{4}{3}tf_1'+\left(\lambda+\tfrac{2}{3}\right)\left(\lambda+\tfrac{3}{3}\right)f_1=b_0\left(\tfrac{3}{2}\right)^{-1/3}(1-t^2)^{1/3}(tf_0'-\lambda f_0) \quad (3.7)$$

$$(1-t^2)f_2''-\tfrac{4}{3}tf_2'+\left(\lambda+\tfrac{4}{3}\right)\left(\lambda+\tfrac{5}{3}\right)f_2=b_0\left(\tfrac{3}{2}\right)^{-1/3}(1-t^2)^{1/3}\left[tf_1'-\left(\lambda+\tfrac{2}{3}\right)f_1\right]+ \\ +b_1\left(\tfrac{3}{2}\right)^{1/3}(1-t^2)^{2/3}(tf_0'-\lambda f_0) \quad (3.8)$$

$$(1-t^2)f_3''-\tfrac{4}{3}tf_3'+\left(\lambda+\tfrac{6}{3}\right)\left(\lambda+\tfrac{7}{3}\right)f_3= \\ =b_0\left(\tfrac{3}{2}\right)^{-1/3}(1-t^2)^{1/3}\left[tf_2'-\left(\lambda+\tfrac{4}{3}\right)f_2\right]+ \\ +b_1\left(\tfrac{3}{2}\right)^{1/3}(1-t^2)^{2/3}\left[tf_1'-\left(\lambda+\tfrac{2}{3}\right)f_1\right]+b_2\left(\tfrac{3}{2}\right)(1-t^2)(tf_0'-\lambda f_0) \quad (3.9)$$

4. The general solution of (3.6) is expressible in terms of hypergeometric functions

$$f_0(t)=A_0F\left(-\tfrac{\lambda}{2},\ \tfrac{\lambda}{2}+\tfrac{1}{6},\ \tfrac{1}{2};\ t^2\right)+B_0tF\left(\tfrac{1}{2}-\tfrac{\lambda}{2},\ \tfrac{2}{3}+\tfrac{\lambda}{2},\ \tfrac{3}{2};\ t^2\right) \quad (4.1)$$

We note that the left-hand side of (3.7) is obtainable from (3.6) by substituting $\lambda + 2/3$ for λ, so that the general integral of (3.7) can be written

$$f_1(t)=f_{01}(t)+\varphi_1(t) \quad (4.2)$$

where the general solution $f_{01}(t)$ of the homogeneous part of (3.7) is representable in the form

$$f_{01}(t)=A_1F\left(-\tfrac{\lambda}{2}-\tfrac{1}{3},\ \tfrac{\lambda}{2}+\tfrac{1}{2},\ \tfrac{1}{2};\ t^2\right)+B_1tF\left(\tfrac{1}{6}-\tfrac{\lambda}{2},\ 1+\tfrac{\lambda}{2},\ \tfrac{3}{2};t^2\right) \quad (4.3)$$

The particular solution $\phi_1(t)$ of the nonhomogeneous equation (3.7) can be expressed

$$\varphi_1(t)=-\tfrac{1}{2}b_0\left(\tfrac{3}{2}\right)^{2/3}(1-t^2)^{1/3}f_0(t) \quad (4.4)$$

as can be easily verified.

In seeking a solution of Equation (3.8), we first express its right-hand side in terms of $f_1(t)$ as given by (4.2) and (4.4):

$$(1-t^2)f_2''-\tfrac{4}{3}tf_2'+\left(\lambda+\tfrac{4}{3}\right)\left(\lambda+\tfrac{5}{3}\right)f_2= \\ =b_0\left(\tfrac{3}{2}\right)^{-1/3}(1-t^2)^{1/3}\left[tf_{01}'-\left(\lambda+\tfrac{2}{3}\right)f_{01}\right]+ \\ +\left(\tfrac{3}{2}\right)^{1/3}\left(b_1-\tfrac{1}{2}b_0^2\right)(1-t^2)^{2/3}(tf_0'-\lambda f_0)+\left(\tfrac{2}{3}\right)^{2/3}\tfrac{1}{2}b_0^2(1-t^2)^{-1/3}f_0 \quad (4.5)$$

Proceeding as before, we can represent the general solution of

(4.5) as

$$f_2(t) = f_{02}(t) + \varphi_{21}(t) + \varphi_{22}(t) \tag{4.6}$$

where $f_{02}(t)$, the general solution of the homogeneous part of (4.5), is given by

$$f_{02}(t) = A_2 F\left(-\frac{\lambda}{2} - \frac{2}{3}, \frac{\lambda}{2} + \frac{5}{6}, \frac{\lambda}{2}; t^2\right) + B_2 t F\left(-\frac{1}{6} - \frac{\lambda}{2}, \frac{4}{3} + \frac{\lambda}{2}, \frac{3}{2}; t^2\right) \tag{4.7}$$

and the particular solutions by

$$\varphi_{21}(t) = -\frac{1}{2} b_0 \left(\frac{3}{2}\right)^{2/3} (1 - t^2)^{1/3} f_{01}(t) \tag{4.8}$$

$$\varphi_{22}(t) = (1 - t^2)^{2/3} [\alpha_2 f_0(t) + \beta_2 t f_0'(t)] \tag{4.9}$$

$$\alpha_2 = \left(\frac{3}{2}\right)^{4/3} \frac{b_0^2 + {}^3/_4 \lambda ({}^1/_2 b_0^2 - b_1)}{4 + 3\lambda}$$

$$\beta_2 = \left(\frac{3}{2}\right)^{4/3} \frac{3 (b_1 + {}^1/_2 b_0^2)}{(4 + 3\lambda)(1 + 3\lambda)}$$

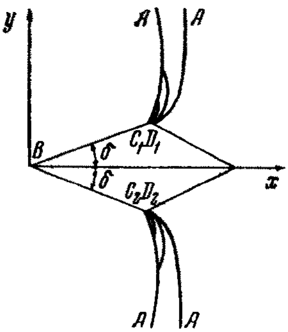

Fig. 1.

In this manner, the solutions of the system (3.5) are reduced to a recurrence procedure which can be carried out without difficulties. If in Equation (2.3) we were to neglect the term with the first derivative, we would be reduced to Tricomi's equation, for which each term of the series (3.4) provides a self-similar solution.

5. Let us consider the flow around a diamond airfoil at zero angle of attack with the free-stream speed equal to the speed of sound at infinity (Fig. 1)*. To the flow regions bounded by the sides of the front wedge, BC_1 and BC_2, and by the limiting characteristics C_1A and C_2A, corresponds the region $BC_1D_1AD_2C_2B$ in the plane of the variables θ, η (Fig. 2). We are led to the following boundary-value problem: find the solution $\psi(\theta, \eta)$ of Equation (2.3) in this region, subject to conditions

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* This problem has been treated by many authors. The first and most significant solution was that of Ovsiannikov [3]. Later it was solved by Guderley and Yoshihara [4] by a different method. In these papers, the problem was simplified by replacing Chaplygin's equation (2.3) by Tricomi's equation (1.1). An attempt to solve the problem in Chaplygin's formulation was made by Aslanov [5], but his solution cannot correspond to physical reality since it becomes infinite on the limiting characteristics.

$$\psi(\delta,\ \eta) = 0 \qquad (5.1)$$

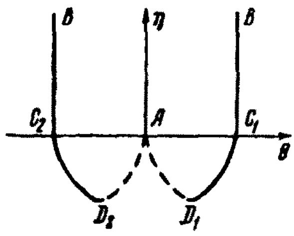

Fig. 2.

$\psi = 0$ on the hodograph characteristics C_1D_1 and C_2D_2 (5.2)

$$\psi(\theta,\ +\infty) = 0 \qquad (5.3)$$

$$\psi(-\theta,\ \eta) = -\psi(\theta,\ \eta) \qquad (5.4)$$

The function $\psi(\theta,\ \eta)$ is bounded on the sonic line C_1C_2 and on the characteristics AD_1 and AD_2, and ψ grows without limit as point A is approached from within the region or along the characteristics AD_1 and AD_2. (5.5)

Next, we tackle the problem of finding a solution of Chaplygin's equation with a singular point on the sonic line.

6. In order to satisfy the condition (5.5), we shall seek a solution of (2.3) in the series form (3.4). For this, we must set $\lambda = -5/3$, as has been shown by Frankl [6]. Hence, the series (3.4) takes the form

$$\psi_0(\theta,\ \eta) = \rho^{-5/3} f_0(t) + \rho^{-1} f_1(t) + \rho^{-1/3} f_2(t) + \rho^{1/3} f_3(t) + \dots \qquad (6.1)$$

Using (4.1) with $\lambda = -5/3$, and setting $A_0 = 0$ so as to satisfy condition (5.4), we obtain

$$\begin{aligned} f_0(t) &= B_0 t F\left(\tfrac{4}{3},\ -\tfrac{1}{6},\ \tfrac{3}{2};\ t^2\right) \\ &= \tfrac{9}{16} B_0 \left[(1-t^2)^{1/3}\left(\tfrac{1}{3}+t\right) - (1+t)^{1/3}\left(\tfrac{1}{3}-t\right)\right] \end{aligned} \qquad (6.2)$$

In seeking the second term of (6.1) which would satisfy condition (5.4), we set $\lambda = -5/3$ and $A_1 = 0$ in (4.3), and with the aid of (4.2) and (4.3) arrive at

$$\rho^{-1} f_1(t) = B_1 \rho^{-1} t F\left(1,\ \tfrac{1}{6},\ \tfrac{3}{2},\ t^2\right) - \tfrac{1}{2} b_0{}^2 \left(\tfrac{3}{2}\right)^{2/3} \rho^{-1} (1-t^2)^{1/6} f_0(t) \qquad (6.3)$$

The behavior of hypergeometric functions indicates that the first term in (6.3) becomes infinite as ρ approaches zero, i.e. on the characteristics AD_1 and AD_2, so that the condition (5.5) requires setting $B_1 = 0$, and (6.3) simplifies to

$$\rho^{-1} f_1(t) = -\tfrac{1}{2} b_0 \eta \rho^{-5/3} f_0(t) \qquad (6.4)$$

In order to find the third term of (6.1), we set $\lambda = -5/3$ and $A_2 = 0$

in (4.7) and find

$$f_{02}(t) = B_2 t F\left(\tfrac{2}{3},\ \tfrac{1}{2},\ \tfrac{3}{2};\ t^2\right) \tag{6.5}$$

With $B_1 = 0$, it follows from (4.8) that $\phi_{21}(t)$ vanishes. Also (4.9) becomes

$$\varphi_{22}(t) = \tfrac{3}{4}\eta^2\rho^{-4/3}\left[\left(b_1 + \tfrac{1}{2}b_1\right) t f_0' - \left(b_0^2 + \tfrac{5}{3}b_1\right) f_0\right] \tag{6.6}$$

Then, by virtue of (4.6) we have for the third term in (6.1)

$$\rho^{-1/3} f_2(t) = B_2\rho^{-1/3} t F\left(\tfrac{2}{2},\ \tfrac{1}{3},\ \tfrac{3}{2};\ t^2\right) + \rho^{-1/3}\varphi_{22}(t) \tag{6.7}$$

The coefficient B_2 must vanish lest the leading term grow indefinitely as $\rho \to 0$, and we find

$$\rho^{-1/3} f_2(t) = \tfrac{3}{4}\eta^2\rho^{-5/3}\left[\left(b_1 + \frac{b_0^2}{2}\right) t f_0' - \left(b_0^2 + \tfrac{5}{3}b_1\right) f_0\right]$$

Substituting for $f_0(t)$ the value from (6.2), we arrive at the final form

$$\rho^{-1/3} f_2(t) = \tfrac{9}{16} B_0\left(\tfrac{3}{2}\right)^{4/3}\left(b_1 + \frac{b_0^2}{2}\right)\rho^{-1/3} t\left[(1+t)^{2/3}\left(\tfrac{2}{3} - t\right) + (1-t)^{2/3}\left(\tfrac{2}{3} + t\right)\right] - \tfrac{3}{4}\left(b_0^2 + \tfrac{5}{3}b_1\right)\eta^2\rho^{-5/3} f_0(t) \tag{6.8}$$

The rest of the terms in the expansion (6.1) are bounded on the characteristics and contain an expression with an arbitrary constant factor, which we choose so as to make $f_k(1) = 0$ for $k > 2$.

In order to satisfy the condition (5.1), it is necessary for the solution to be periodic in θ with the period of 2δ. In order to make the solution (6.1) periodic in θ, let us focus on its evaluation on the sonic line, i.e. for $\eta = 0$, $t = 1$ and $\rho = \theta$

$$\psi_0(\theta,\ 0) = f_0(1)\,\theta^{-5/3} + f_1(1)\,\theta^{-1} + f_2(1)\,\theta^{-1/3}$$

By virtue of (6.2) and (6.8) we are led to

$$f_0(1) = \tfrac{3}{8}\, 2^{1/3} B_0,\ f_1(1) = 0,\ \ f_2(1) = -\tfrac{9}{32} B_0 6^{1/3}\left(b_1 + \tfrac{1}{2} b_0^2\right)$$

$$\psi_0(\theta,\ 0) = \tfrac{3}{8}\, 2^{1/3} B_0\left[\theta^{-5/3} - \tfrac{3}{4}\, 3^{1/3}\left(b_1 + \tfrac{1}{2} b_0^2\right)\theta^{-1/3}\right] \tag{6.9}$$

7. Let us turn to Equation (2.3) and seek its solutions in the special form $\psi_n(\theta,\ \eta) = s_n(\eta)\sin(2\pi n\theta/\delta)$. Such solutions satisfy boundary condition (5.1) for integral values of the index n. Substituting the expression ψ_n into (2.3), we obtain an equation for $s_n(\eta)$:

$$\frac{d^2 s_n}{d\eta^2} + b(\eta)\frac{ds_n}{d\eta} - \frac{\pi^2 n^2}{\delta^2} s_n = 0 \tag{7.1}$$

If we require that $s_n(+\infty) = 0$ and $s_n(0) = 1$, the integral of (7.1) is uniquely determined. By virtue of (2.1) and (2.3) such an integral can be expressed explicitly in terms of the hypergeometric functions of Chaplygin

$$s_n(\eta) = \frac{z_\nu(\tau)}{z_\nu(\tau_*)}, \quad z_\nu(\tau) = \tau^\nu F(a_\nu,\ b_\nu;\ 2\nu + 1;\ \tau) \tag{7.2}$$

Here

$$\nu = \frac{\pi n}{2\delta}, \qquad a_\nu + b_\nu = 2\nu - \beta, \qquad a_\nu b_\nu = -\beta\nu(2\nu + 1), \qquad \tau_* = \frac{1}{2\beta + 1}$$

The convergent infinite series

$$\psi(\theta,\ \eta) = \sum_{n=1}^{\infty} A_n s_n(\eta) \sin\frac{\pi n}{\delta}\theta \tag{7.3}$$

represents an exact solution of (2.3), satisfying the boundary conditions (5.1), (5.3) and (5.4). In order to satisfy the essential condition (5.5), it is necessary to choose coefficients A_n in (7.3) in such a way that, for $\eta = 0$ and $\theta \to 0$, the evaluation of the series (7.3) would match Expression (6.9). Then the series (7.3) will represent a function bounded on the characteristics AD_1 and AD_2.

8. Next, we need to study the following special solution of Chaplygin's equation (2.1):

$$F_\lambda(\theta,\ \tau) = \sum_{n=1}^{\infty} \frac{1}{n^\lambda} \frac{z_{n/2}(\tau)}{z_{n/2}(\tau_*)} \sin n\theta \tag{8.1}$$

By virtue of the properties of the functions $z_{n/2}(\tau)$, the series (8.1) converges for $\tau < \tau_*$ for arbitrary values of the parameter λ. In order to evaluate the series for $\tau = \tau_*$, we utilize a formula from the theory of gamma functions:

$$\frac{1}{n^\lambda} = \frac{1}{\Gamma(\lambda)\,(e^{2\pi i\lambda} - 1)} \int_{\infty}^{(0+)} t^{\lambda - 1} e^{-nt}\,dt \tag{8.2}$$

Here the integration is carried out along a contour which follows the positive real axis first above and then below, and circles the origin in a counter-clockwise direction.

Substituting (8.2) into (8.1), we obtain

$$F_\lambda(\theta,\ \tau) = \frac{1}{\Gamma(\lambda)\,(e^{2\pi i\lambda} - 1)} \int_{\infty}^{(0+)} t^{\lambda-1} \left(\sum_{n=1}^{\infty} \frac{z_{n/2}(\tau)}{z_{n/2}(\tau_*)}\, e^{-nt} \sin n\theta \right) dt \tag{8.3}$$

which, on the sonic line $\tau = \tau_*$, becomes

$$F_\lambda(\theta,\ \tau_*) = \frac{1}{\Gamma(\lambda)\,(e^{2\pi i\lambda} - 1)} \int_{\infty}^{(0+)} \frac{t^{\lambda-1}\sin\theta}{\cosh t - \cos\theta}\, dt \tag{8.4}$$

In (8.4), for small values of θ, the poles of the integrand, at $t = \pm i\theta$, lie close to the path of integration. Taking this fact into account, it is easy to find the representation of $F_\lambda(\theta,\ \tau_*)$ as $\theta \to 0$:

$$F_\lambda(\theta,\ \tau_*) \sim \Gamma(1 - \lambda) \cos \frac{\pi\lambda}{2} |\theta|^{\lambda-1} \tag{8.5}$$

In particular, for $\lambda = -2/3$ and $\lambda = 2/3$, we have, respectively

$$F_{-2/3}(\theta,\tau_*) \sim \Gamma\left(\tfrac{5}{3}\right) \cos \tfrac{1}{3}\pi\theta^{-5/3}, \qquad F_{2/3}(\theta,\ \tau_*) \sim \Gamma\left(\tfrac{1}{3}\right) \cos \tfrac{1}{3}\pi\theta^{-1/3} \tag{8.6}$$

Consequently, we choose for the coefficients of the series (7.3)

$$A_n = C\,(n^{2/3} - a n^{-2/3}) \tag{8.7}$$

where C is an arbitrary constant, and

$$a = \frac{3^{1/3}\Gamma(2/3)\,\delta^{4/3}}{2\pi^{4/3}\,\Gamma(1/3)} \left(b_1 + \frac{b_0^2}{2} \right) \tag{8.8}$$

In other words, the expression

$$\psi(\theta,\ \tau) = C \sum_{n=1}^{\infty} \left(n^{2/3} - \frac{a}{n^{2/3}} \right) \frac{z_\nu(\tau)}{z_\nu(\tau_*)} \sin 2\nu\theta \qquad \left(\nu = \frac{\pi n}{2\delta} \right) \tag{8.9}$$

satisfies all the required boundary conditions except (5.2). This remaining condition can be met by adding to the series (8.9) the following series:

$$\sum_{n=1}^{\infty} \alpha_n \frac{z_\nu(\tau)}{z_\nu(\tau_*)} \sin 2\nu\theta$$

In satisfying condition (5.2) at separate points of the characteristics C_1D_1 and C_2D_2 (see Fig. 2), we arrive at a system of linear equations, which determine a_n. The actual values of a_n turn out to be small compared to the coefficients of series (8.9), so that they may be neglected. Thus, the exact solution of Chaplygin's equation given by (8.9) does not vanish on the characteristics C_1D_1 and C_2D_2 (see Fig. 2), but remains small.

9. Utilizing Expression (8.9) for the stream-function ψ, we can obtain formulas for the velocity along the front wedge. As shown by Chaplygin, the physical abscissa is related to the potential function ϕ and to ψ according to

$$dx = \frac{\cos\theta}{v}\,d\varphi - \frac{\sin\theta}{(1-\tau)^{\beta}}\,d\psi \tag{9.1}$$

and these two functions are interrelated as follows:

$$\frac{\partial\varphi}{\partial\theta} = \frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial\psi}{\partial\tau}, \qquad \frac{\partial\varphi}{\partial\tau} = -\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial\psi}{\partial\theta} \tag{9.2}$$

Since along the top of the wedge $\theta = \delta$ and $d\psi = 0$, utilization of the second of equations (9.2) leads to

$$x = -\frac{\cos\delta}{2v_{\max}}\int_0^{\tau}\frac{1-(2\beta+1)\tau}{\tau\sqrt{\tau}\,(1-\tau)^{\beta+1}}\left(\frac{\partial\psi}{\partial\theta}\right)_{\theta=\delta}d\tau \tag{9.3}$$

From the solution (8.9) we derive

$$\left(\frac{\partial\psi}{\partial\theta}\right)_{\theta=\delta} = C\sum_{n=1}^{\infty}\frac{\pi}{\delta}(-1)^n\left(n^{5/3}-an^{1/3}\right)\frac{z_\nu(\tau)}{z_\nu(\tau_*)}$$

Substituting this into (9.3) and evaluating the integral, we find

$$x(\tau) = \frac{C\pi\cos\delta}{\delta v_{\max}\sqrt{\tau}\,(1-\tau)^{\beta}}\sum_{n=1}^{\infty}(-1)^n\frac{n^{5/3}-an^{1/3}}{\pi^2n^2/\delta^2-1}\,\frac{z_\nu(\tau)+2\tau z_\nu'(\tau)}{z_\nu(\tau_*)} \tag{9.4}$$

This relation determines the velocity along the wedge. In order to establish the value of the constant C we note that

$$\lim_{\tau\to\tau_*}\frac{x(\tau)}{\cos\delta} = l \qquad \text{as } \tau\to\tau_*$$

where l is the length of the wedge; hence

$$l = \frac{C\pi}{\delta v_{\max}\sqrt{\tau_*}\,(1-\tau_*)^{\beta}}\lim_{\tau\to\tau_*}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n^{5/3}-an^{1/3}\right)}{n^2\pi^2/\delta^2-1}\,\frac{z_\nu(\tau)+2\tau z_\nu'(\tau)}{z_\nu(\tau_*)} \tag{9.5}$$

It is important to note that in (9.5) it is not permissible to interchange the limiting process $\tau\to\tau_*$ and the summing process because the resulting series diverges. For the sake of brevity let us designate

$$S_1 = \lim_{\tau\to\tau_*}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(n^{5/3}-an^{1/3}\right)}{n^2\pi^2/\delta^2-1}\,\frac{z_\nu(\tau)+2\tau z_\nu'(\tau)}{z_\nu(\tau_*)} \tag{9.10}$$

and rewrite (9.5) as

$$l = \frac{C\pi}{\delta v_{\max} \sqrt{\tau_*}(1-\tau_*)^{\beta}} S_1 \tag{9.11}$$

10. The expression for the coefficient of pressure for the case of sonic free-stream velocity reads:

$$C_p = \frac{p - p_*}{{}^1\!/_2 \rho_* a_*^2} = \frac{2}{\kappa}\left[\left(\frac{1-\tau}{1-\tau_*}\right)^{\beta+1} - 1\right] \tag{10.1}$$

By integrating along the front wedge we find the drag coefficient:

$$C_x = \frac{2\tan\delta}{l}\int_0^l C_p dx = \frac{2\tan\delta}{l}\int_0^{\tau_*} C_p \frac{dx}{d\tau} d\tau = -\frac{2\tan\delta}{l}\int_0^{\tau_*} x\frac{dC_p}{d\tau} d\tau \tag{10.2}$$

Upon substitution of the expressions for x and $dC_p/d\tau$ from (9.4) and (10.1), we are led to

$$C_x = \frac{8\pi c \sin\delta}{\delta l(\varkappa-1)(1-\tau_*)^{\beta+1} v_{\max}} \int_0^{\tau_*} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{5/3} - an^{1/3}}{n^2\pi^2/\delta^2 - 1} \left(\frac{\sqrt{\tau} z_\nu'(\tau)}{z_\nu(\tau_*)}\right)^{\bullet} d\tau$$

Evaluating the integral, we finally arrive at

$$C_x = \frac{8\pi c \sin\delta \sqrt{\tau_*}}{l\delta(\varkappa-1)(1-\tau_*)^{\beta+1} v_{\max}} S_2 \tag{10.3}$$

where

$$S_2 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{5/3} - an^{1/3}}{n^2\pi^2/\delta^2 - 1} \tag{10.4}$$

Eliminating the constant C from (10.3) with the aid of (9.11), we obtain

$$C_x' = 4\sin\delta \frac{S_2}{S_1} \tag{10.5}$$

11. In order to evaluate the limit S_1 in (9.10), let us rewrite it in the form

$$S_1 = \lim_{\tau\to\tau_*} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^{5/3} - an^{1/3})}{n^2\pi^2/\delta^2 - 1} \frac{z_\nu(\tau)}{z_\nu(\tau_*)} + \lim_{\tau\to\tau_*} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^{5/3} - an^{1/3})}{n^2\pi^2/\delta^2 - 1} \frac{2\tau z_\nu'(\tau)}{z_\nu(\tau_*)}$$

or $S_1 = S_2 + S_3$, where

$$S_3 = \lim_{\tau\to\tau_*} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^{5/3} - an^{1/3})}{n^2\pi^2/\delta^2 - 1} \frac{2\tau z_\nu'(\tau)}{z_\nu(\tau_*)} \tag{11.1}$$

In the evaluation of S_3 we use the asymptotic formula [7]

$$\frac{2\tau_* z_\nu'(\tau_*)}{z_\nu(\tau_*)} = A_1\nu^{2/3} + A_2 + \frac{A_3}{\nu^{2/3}} + O\left(\frac{1}{\nu^{4/3}}\right)$$

$$A_1 = \frac{2^{2/3}3^{1/3}\Gamma(2/3)}{\Gamma(1/3)}(\varkappa+1)^{1/3}, \qquad A_2 = -\frac{2\varkappa+5}{10}$$

and transform (11.1) as follows:

$$S_3 = \lim_{\tau\to\tau_*}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n^{8/3}-an^{1/3})}{n^2\pi^2/\delta^2-1}\left(\frac{2\tau z_\nu'(\tau)}{z_\nu(\tau_*)} - A_1\nu^{2/3}\left(\frac{\tau}{\tau_*}\right)^n\right) +$$

$$+\lim_{\tau\to\tau_*}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n^{5/3}-an^{2/3})}{n^2\pi^2/\delta^2-1}A_1 n^{2/3}\left(\frac{\pi}{2\delta}\right)^{2/3}\left(\frac{\tau}{\tau_*}\right)^n \qquad \left(\nu=\frac{\pi n}{2\delta}\right) \quad (11.2)$$

In proceeding to the limit in the first summation, we find a convergent series; denoting it by S_4, we have $S_3 = S_4 + S_5$, where now

$$S_5 = \lim_{\tau\to\tau_*}A_1\left(\frac{\pi}{2\delta}\right)^{2/3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n^{7/3}-an)}{n^2\pi^2/\delta^2-1}\left(\frac{\tau}{\tau_*}\right)^n \qquad (11.3)$$

This expression can be rewritten in the form

$$S_5 = \frac{A_1}{2^{2/3}}\left(\frac{\delta}{\pi}\right)^{4/3}\lim_{\tau\to\tau_*}\sum_{n=1}^{\infty}\frac{(-1)^n\, n^{1/3}}{1-\delta^2/\pi^2n^2}\left(\frac{\tau}{\tau_*}\right)^n - \frac{A_1 a}{2^{2/3}}\left(\frac{\pi}{\delta}\right)^{2/3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,n}{n^2\pi^2/\delta^2-1}$$

however

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^{1/3}}{1-\delta^2/\pi^2n^2}\left(\frac{\tau}{\tau_*}\right)^n = \sum_{n=0}^{\infty}\left(\frac{\delta}{\pi}\right)^{2k}\sum_{k=1}^{\infty}\frac{(-1)^{n+1}}{n^{2k-1/3}}\left(\frac{\tau}{\tau_*}\right)^n$$

According to the well-known properties of the ζ-function [10], we have

$$\lim_{\tau\to\tau_*}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{2k-1/3}}\left(\frac{\tau}{\tau_*}\right)^n = (1-2^{4/3-2k})\,\zeta\left(2k-\frac{1}{3}\right)$$

In this manner we finally arrive at

$$S_5 = \frac{A_1}{2^{2/3}}\sum_{k=0}^{\infty}(1-2^{4/3-2k})\,\zeta\left(2k-\frac{1}{3}\right)\left(\frac{\delta}{\pi}\right)^{2k+4/3} - \frac{A_1 a}{2^{2/3}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2\pi^2/\delta^2-1}. \qquad (11.4)$$

All the investigated sums can be represented in the form of power series in the variable δ/π. A similar representation can be carried out for the drag coefficient C_x. Detailed calculations of C_x for different angles δ will be presented in a separate paper.

12. Gas flow around a flat plate at sonic speed. Let us consider a plate of length l (or an airfoil with a flat lower surface) at an angle of attack δ in a stream which has a sonic speed $v = a_*$ (Fig. 3).

The flow field is represented in the hodograph plane in Fig. 4. All streamlines issue from the point A, which corresponds to the flow at infinity, and which therefore appears as a singular point.

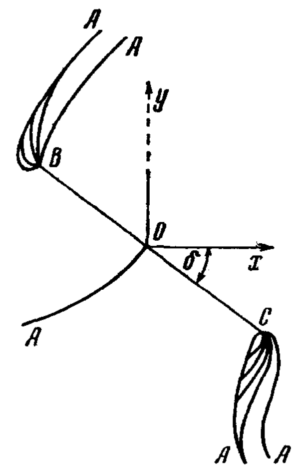

Fig. 3.

The problem reduces to the determination of the stream function ψ satisfying the Chaplygin equation (2.1) in the region $AB'BOCC'A$, which possesses a singularity at A and vanishes on part of the boundary, namely $C'COBB'$ (see Fig. 4). The solution of this problem within the framework of the Tricomi equation has been obtained by Guderley [8].

Following the method already presented, we construct a solution of Chaplygin's equation, which satisfies all the required boundary conditions except the vanishing of ψ on BB' and CC'. Letting $\delta = \pi$ in (8.9), we obtain a special integral of (2.3) in the form

$$\psi_1(\theta, \tau) = c \sum_{n=1}^{\infty} \left(n^{2/3} - \frac{a}{n^{2/3}}\right) \frac{z_{n/2}(\tau)}{z_{n/2}(\tau_*)} \sin n\theta \qquad (12.1)$$

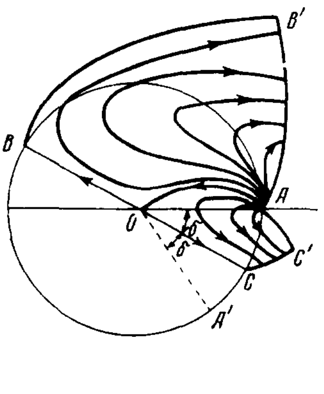

Fig. 4.

To this expression it is necessary to add a function $\psi_2(\theta, \pi)$, even in the variable θ, so that ultimately the flow around the flat plate at an angle δ may be found. Considerations analogous to those used in the preceding development of (2.1) lead to the expression

$$\psi_2 = c \sum_{n=1}^{\infty} \frac{\gamma}{n^{1/3}} \frac{z_{n/2}(\tau)}{z_{n/2}(\tau_*)} \cos n\theta \quad (c,\ \gamma = \text{const}) \qquad (12.2)$$

Superposing the solutions ψ_1 and ψ_2, we find the integral

$$\psi_0 = \psi_1 + \psi_2 = c \sum_{n=1}^{\infty} \left[\left(n^{2/3} - \frac{a}{n^{2/3}}\right) \sin n\theta + \frac{\gamma}{n^{1/3}} \cos n\theta\right] \frac{z_{n/2}(\tau)}{z_{n/2}(\tau_*)} \qquad (12.3)$$

which has the required singularity at the point A. In order to satisfy the conditions of the flow geometry, we add to ψ_0 an analogous expression, corresponding to a singularity at A' (see Fig. 4), a point which is a mirror image of A with respect to the segment BC:

$$\bar{\psi}_0(\tau, \theta) = c\sum_{n=1}^{\infty}\left[\left(n^{2/3} - \frac{a}{n^{2/3}}\right)\sin n(\theta + 2\delta) - \frac{\gamma}{n^{1/3}}\; n \;(\theta + 2\delta)\right]\frac{z_{n/2}(\tau)}{z_{n/2}(\tau_*)}.$$

Then $\psi = \psi_0 + \bar{\psi}_0$ will appear as the desired solution:

$$\psi = \psi_0 + \bar{\psi}_0 = c\sum_{n=1}^{\infty}\left[\left(n^{2/3} - \frac{a}{n^{2/3}}\right)\cos n\delta + \frac{\gamma}{n^{1/3}}\sin n\delta\right]\frac{z_{n/2}(\tau)}{z_{n/2}(\tau_*)}\sin n(\theta + \delta) \tag{12.4}$$

The constant γ is determined from the condition that O (Fig. 3) must be a branch point of the stream function [9]. For this it is clearly necessary for the coefficient of the first term of the series (12.4) to vanish so that

$$(1 - a)\cos\delta + \gamma\sin\delta = 0$$

Consequently

$$\gamma = (a - 1)\cot\delta.$$

Substituting this value of γ into (12.4), we find

$$\psi = c\sum_{n=2}^{\infty}\left[\left(n^{2/3} - \frac{a}{n^{2/3}}\right)\sin\delta\cos n\delta + \frac{a-1}{n^{1/3}}\cos\delta\sin n\delta\right]\frac{z_{n/2}(\tau)}{z_{n/2}(\tau_*)}\sin n(\theta + \delta) \tag{12.5}$$

This solution does not vanish on the characteristics BB' and CC', as required, but its values there are small. The constant c can be expressed in terms of the length of the plate just as in the case of the wedge.

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